

# $K_{l3}$ FORM FACTOR WITH $N_f = 2 + 1$ DOMAIN WALL FERMIONS

J A M E S   Z A N O T T I  
U N I V E R S I T Y   O F   E D I N B U R G H  
U K Q C D / R B C   C O L L A B O R A T I O N S

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# MOTIVATION

- ◆  $K \rightarrow \pi l \nu$  ( $K_{l3}$ ) decay leads to determination of  $|V_{us}|$

decayrate  $\propto |V_{us}|^2 |f_+(q^2 = 0)|^2$

- ◆ Require precise theoretical determination  $f_+(0)$
- ◆ Current conservation  $\Rightarrow f_+(0) = 1 \Big|_{su(3) \text{ flavour limit}}$
- ◆ Ademollo-Gatto Theorem -> second order SU(3) breaking effects in  $f_+(0)$

$$\begin{aligned} f_+(0) &= 1 + f_2 + f_4 + \dots \\ \Rightarrow \Delta f &= 1 + f_2 - f_+(0) \end{aligned}$$

- ◆ [Leutwyler & Roos:  $f_2 = -0.023$ ]

# MOTIVATION

$$\Delta f = 1 + f_2 - f_+(0)$$

-0.016(8) (Leutwyler & Roos, 1984)

-0.017(5)(7) (Bećirević et al., quenched)

$$\Delta f = -0.009(9) \quad (\text{Dawson et al., } N_f = 2 \text{ DWF})$$

-0.025(4) (Tsutsui et al.,  $N_f = 2$  Clover)

-0.016(5) (Preliminary RBC/UKQCD)

Improve on earlier studies by:

- Using  $N_f = 2 + 1$  flavours of dynamical fermions
- Probing light quark masses
- Checking finite size effects

# LATTICE TECHNIQUES

$K \rightarrow \pi$  matrix element

$$\langle \pi(p') | V_\mu | K(p) \rangle = (p_\mu + p'_\mu) f_+(q^2) + (p_\mu - p'_\mu) f_-(q^2), \quad q^2 = (p' - p)^2$$

Three-point function

$$C_\mu^{PQ}(t', t, \vec{p}', \vec{p}) = \sum_{\vec{x}, \vec{y}} e^{-i\vec{p}'(\vec{y}-\vec{x})} e^{-i\vec{p}\vec{x}} \langle 0 | \mathcal{O}_Q(t') | Q(p') \rangle \langle Q(p') | V_\mu(t) | P(p) \rangle \langle P(p) | \mathcal{O}_P^\dagger(0) | 0 \rangle$$

# EXTRACTION OF FORM FACTOR

[RBC: hep-ph/0607162]

Extract scalar form factor

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{m_K^2 - m_\pi^2} f_-(q^2)$$

at  $q_{\max}^2 = (m_K - m_\pi)^2$  with high precision via

$$\begin{aligned} R(t', t) &= \frac{C_4^{K\pi}(t', t; \vec{0}, \vec{0}) C_4^{\pi K}(t', t; \vec{0}, \vec{0})}{C_4^{KK}(t', t; \vec{0}, \vec{0}) C_4^{\pi\pi}(t', t; \vec{0}, \vec{0})} \\ &\longrightarrow \frac{(m_K + m_\pi)^2}{4m_K m_\pi} |f_0(q_{\max}^2)|^2 \end{aligned}$$

# $Q^2$ DEPENDENCE

Construct second ratio

$$\begin{aligned}\tilde{R}(t', t; \vec{p}', \vec{p}) &= \frac{C_4^{K\pi}(t', t; \vec{p}', \vec{p}) C^K(t; \vec{0}) C^\pi(t' - t; \vec{0})}{C_4^{K\pi}(t', t; \vec{0}, \vec{0}) C^K(t; \vec{p}) C^\pi(t' - t; \vec{p}')} \\ &\longrightarrow \frac{(E_K(\vec{p}) + E_\pi(\vec{p}'))^2}{m_K + m_\pi} F(p', p)\end{aligned}$$

where

$$F(p', p) = \frac{f_+(q^2)}{f_0(q_{\max}^2)} \left( 1 + \frac{E_K(\vec{p}) - E_\pi(\vec{p}')} {E_K(\vec{p}) + E_\pi(\vec{p}')} \xi(q^2) \right), \quad \xi(q^2) = \frac{f_-(q^2)}{f_+(q^2)}$$

# $Q^2$ DEPENDENCE

Construct third ratio

$$R_k(t', t; \vec{p}', \vec{p}) = \frac{C_k^{K\pi}(t', t; \vec{p}', \vec{p}) C_4^{KK}(t', t; \vec{p}', \vec{p})}{C_4^{K\pi}(t', t; \vec{p}', \vec{p}) C_k^{KK}(t', t; \vec{p}', \vec{p})} \quad (k = 1, 2, 3)$$

to obtain

$$\xi(q^2) = \frac{-(E_K(\vec{p}) + E_K(\vec{p}'))(p + p')_k + (E_K(\vec{p}) + E_\pi(\vec{p}'))(p + p')_k R_k}{(E_K(\vec{p}) + E_K(\vec{p}'))(p - p')_k - (E_K(\vec{p}) - E_\pi(\vec{p}'))(p + p')_k R_k}$$

# PARAMETERS

- ✿  $N_f = 2 + 1$  flavours of dynamical domain wall fermions
- ✿ Iwasaki gauge action

$$\beta = 2.13, \ L_s = 16, \ am_{\text{res}} \approx 0.03, \ a \approx 0.121 \text{ fm}, \ am_s = 0.04$$

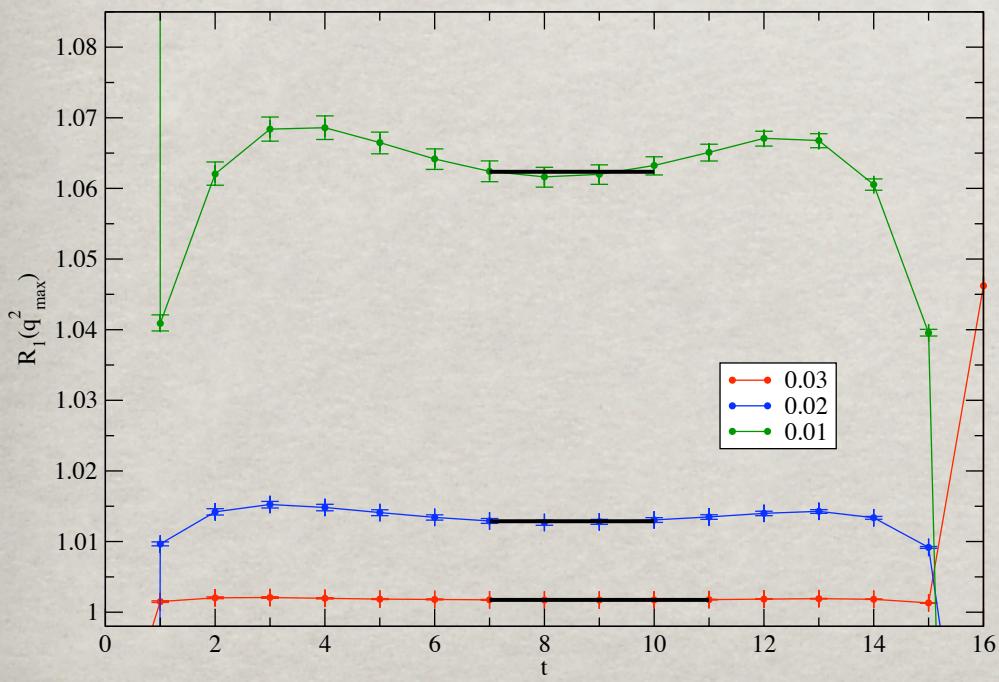
$am_q$	Volume	$m_\pi$ [MeV]	$m_K$ [MeV]
0.03	$16^3 \times 32$	0.624(2)	0.668(2)
0.02		0.517(3)	0.617(2)
0.01		0.393(2)	0.567(1)
0.03	$24^3 \times 64$	0.624(1)	0.665(1)
0.02		0.516(1)	0.614(1)
0.01		0.385(1)	0.559(1)

For more details, see [hep-lat/0701013](https://arxiv.org/abs/hep-lat/0701013)

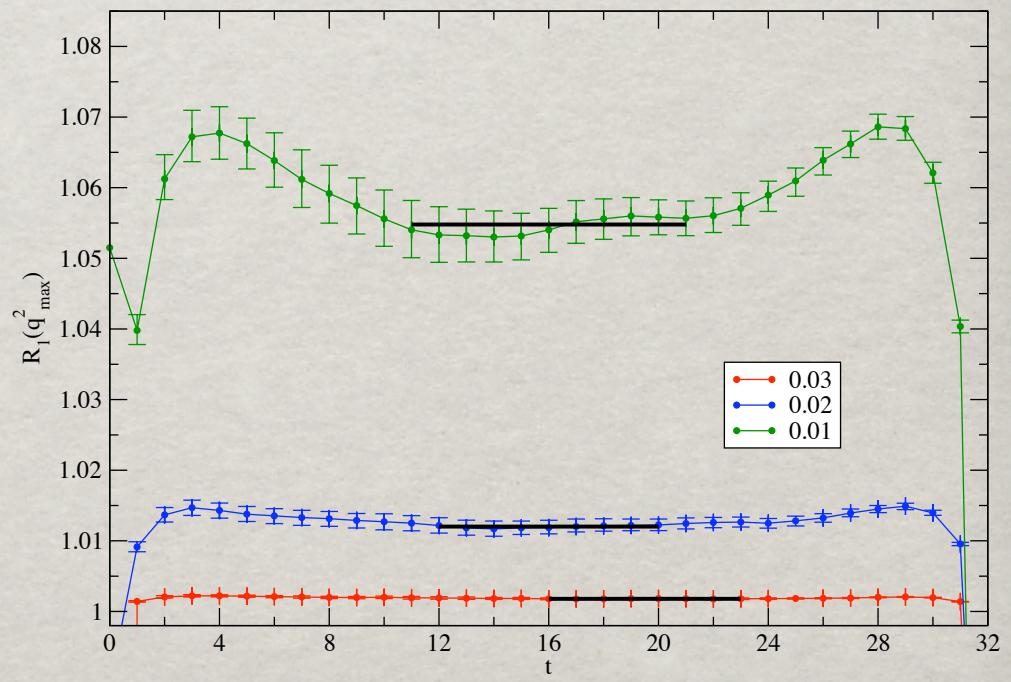
$$f_0(q_{\max}^2), \ am_s = 0.04$$

$$R(t', t) = \frac{C_4^{K\pi}(t', t; \vec{0}, \vec{0}) C_4^{\pi K}(t', t; \vec{0}, \vec{0})}{C_4^{KK}(t', t; \vec{0}, \vec{0}) C_4^{\pi\pi}(t', t; \vec{0}, \vec{0})}$$

$16^3 \times 32$



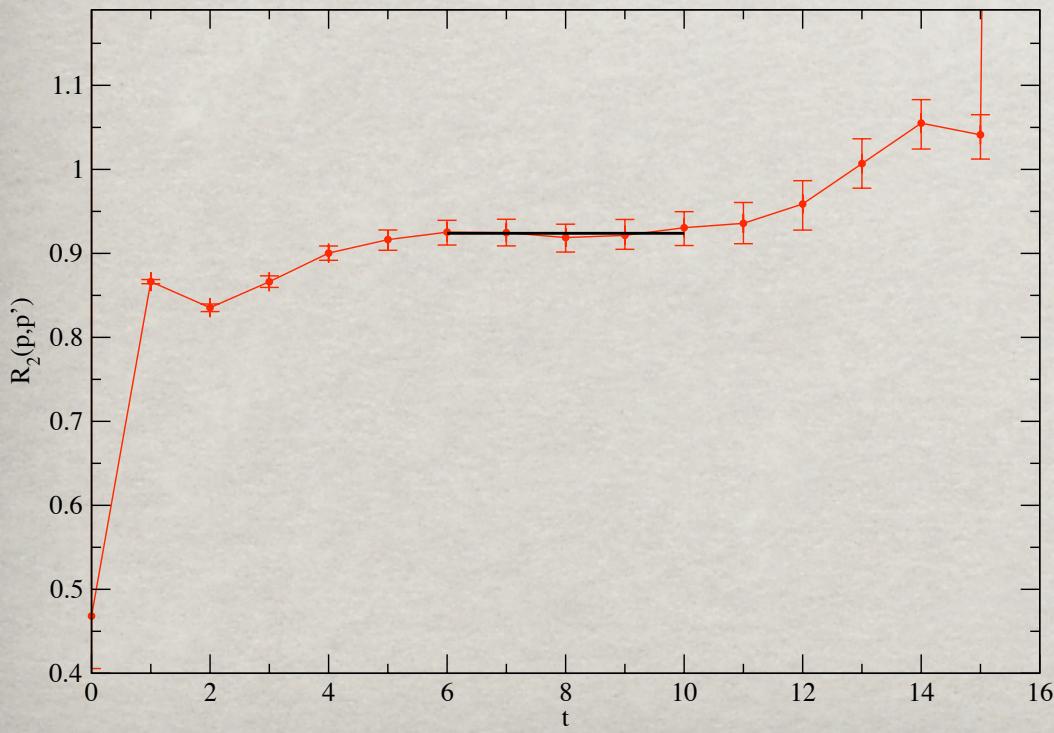
$24^3 \times 64$



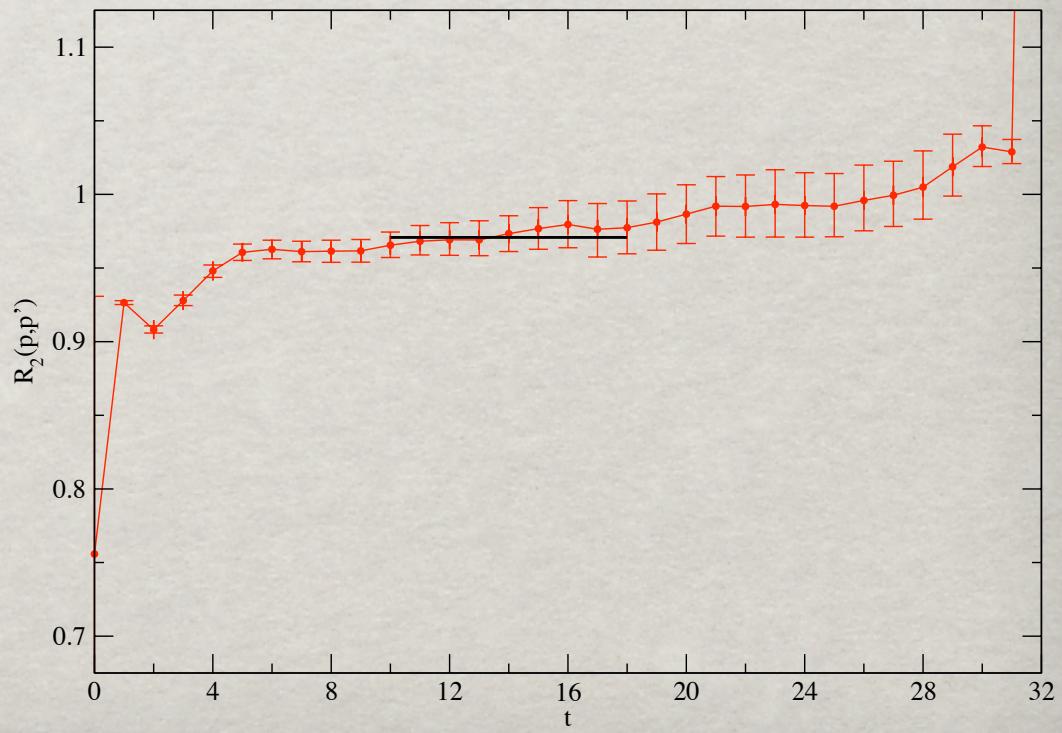
$F(p, p')$ ,  $am_q = 0.02$ ,  $|\vec{q}|^2 = 1$ :

$$\tilde{R}(t', t; \vec{p}', \vec{p}) = \frac{C_4^{K\pi}(t', t; \vec{p}', \vec{p}) C^K(t; \vec{0}) C^\pi(t' - t; \vec{0})}{C_4^{K\pi}(t', t; \vec{0}, \vec{0}) C^K(t; \vec{p}) C^\pi(t' - t; \vec{p}')}$$

$16^3 \times 32$



$24^3 \times 64$

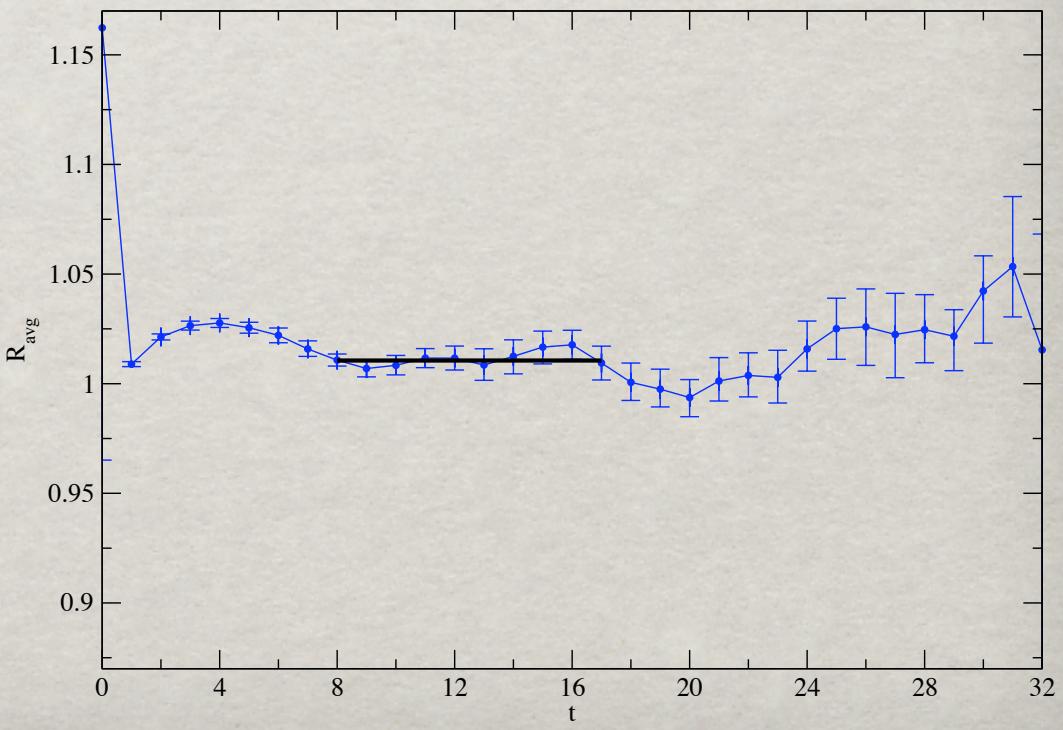
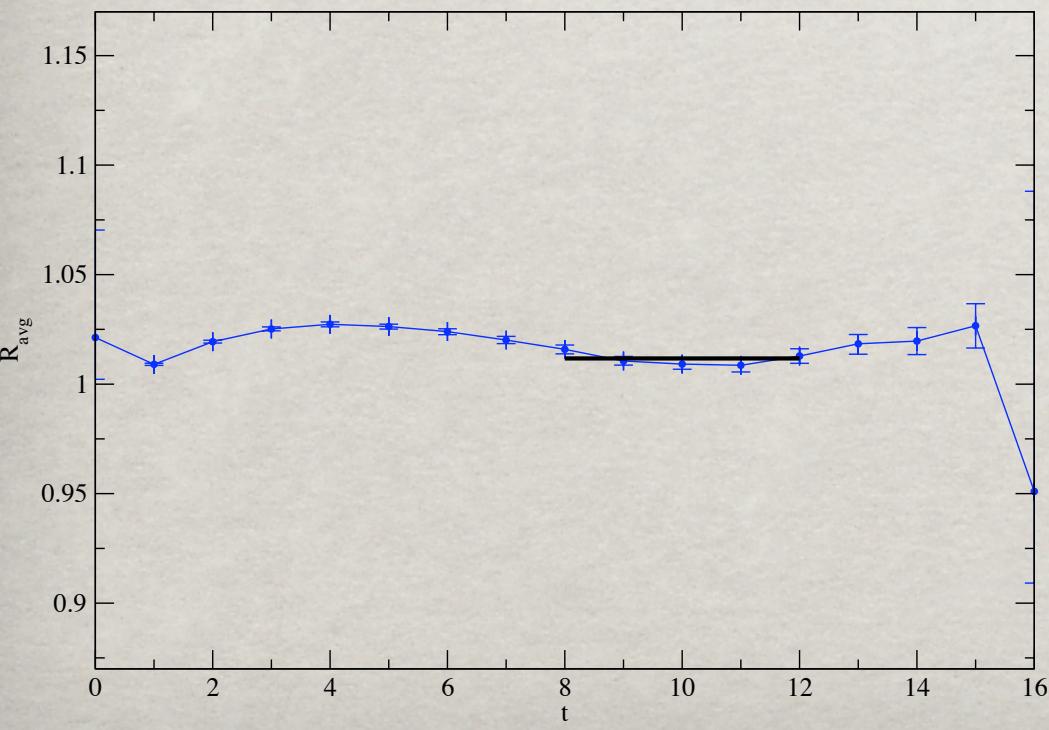


$\xi(q^2), \ am_q = 0.03, \ |\vec{q}|^2 = 2 :$

$$R_k(t', t; \vec{p}', \vec{p}) = \frac{C_k^{K\pi}(t', t; \vec{p}', \vec{p}) C_4^{KK}(t', t; \vec{p}', \vec{p})}{C_4^{K\pi}(t', t; \vec{p}', \vec{p}) C_k^{KK}(t', t; \vec{p}', \vec{p})} \quad (k = 1, 2, 3)$$

$16^3 \times 32$

$24^3 \times 64$



# FITTING FORM FACTORS

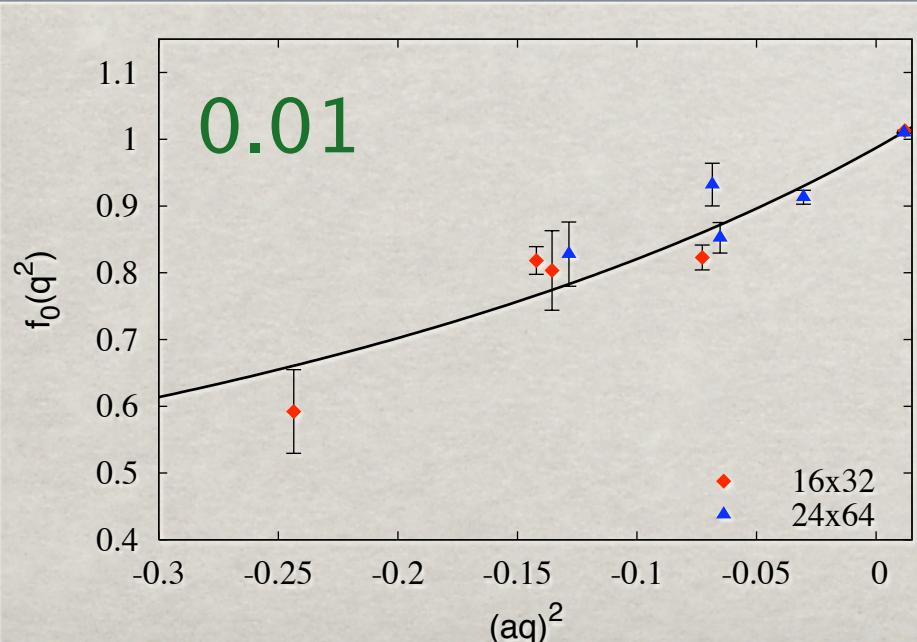
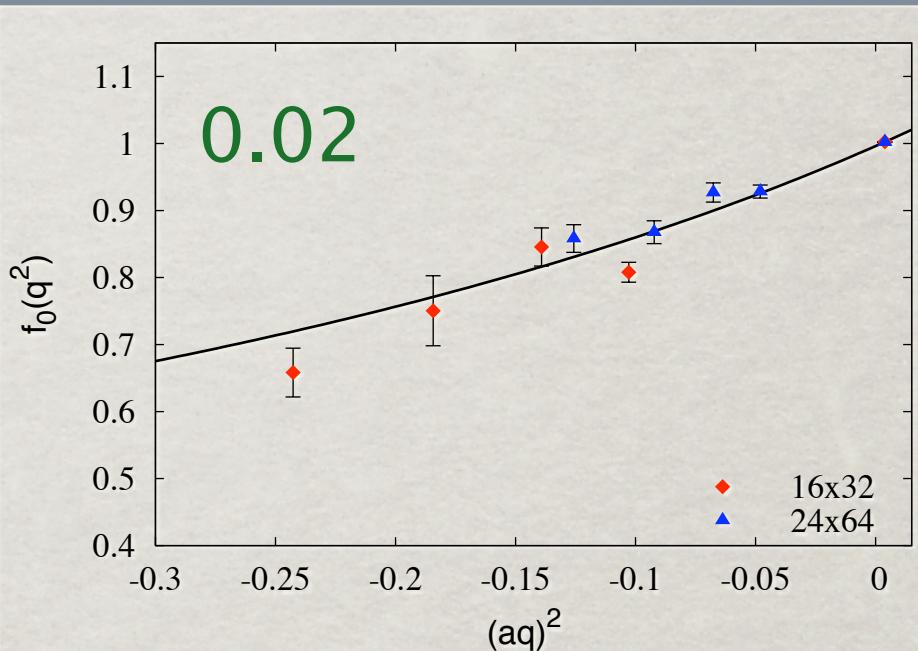
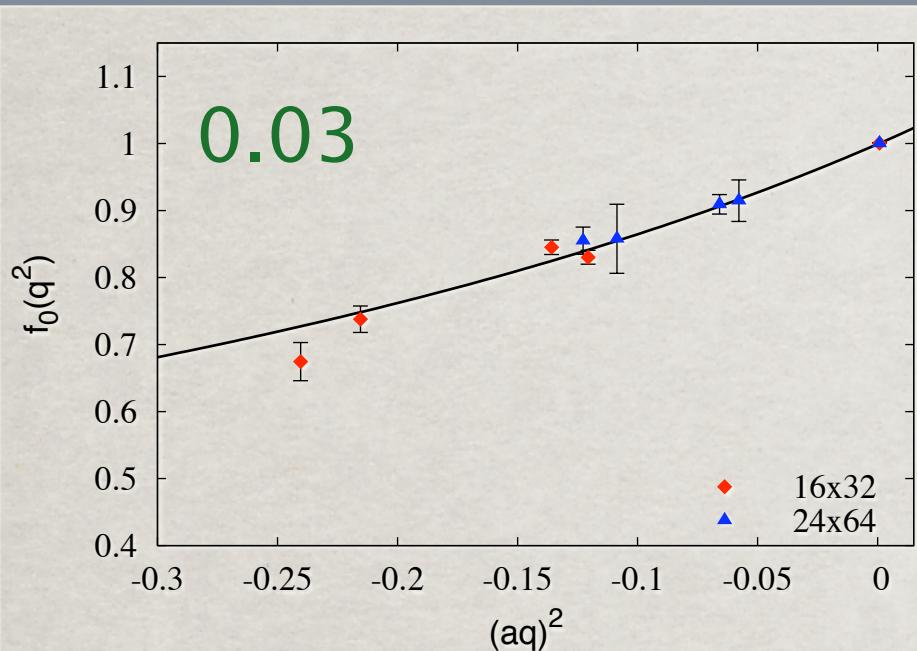
- ✿ Construct scalar form factor:

$$f_0(q^2) = f_+(q^2) \left[ 1 + \frac{q^2}{m_K^2 - m_\pi^2} \xi(q^2) \right]$$

- ✿ Fit with a monopole ansatz:

$$f_0(q^2) = \frac{f_0(0)}{1 - q^2/M^2}$$

**POLE:**  $f_0(q^2) = f_0(0)/(1 - q^2/M^2)$



- 0.03:  $f_0(0) = 0.99911(6)$
- 0.02:  $f_0(0) = 0.99622(51)$
- 0.01:  $f_0(0) = 0.98725(272)$

# CHIRAL EXTRAPOLATION OF $f_+(0)$

$$f_+(0) = 1 + f_2 + \Delta f$$

$$f_2 = \frac{3}{2} H_{\pi K} + \frac{3}{2} H_{\eta K}$$

where

$$H_{PQ} = -\frac{1}{64\pi^2 f_\pi^2} \left[ M_P^2 + M_Q^2 + \frac{2M_P^2 M_Q^2}{M_P^2 - M_Q^2} \log \left( \frac{M_Q^2}{M_P^2} \right) \right]$$

at the physical masses,  $f_2 = -0.023$

$\Delta f \propto (m_s - m_{ud})^2$   Attempt two different extrapolations

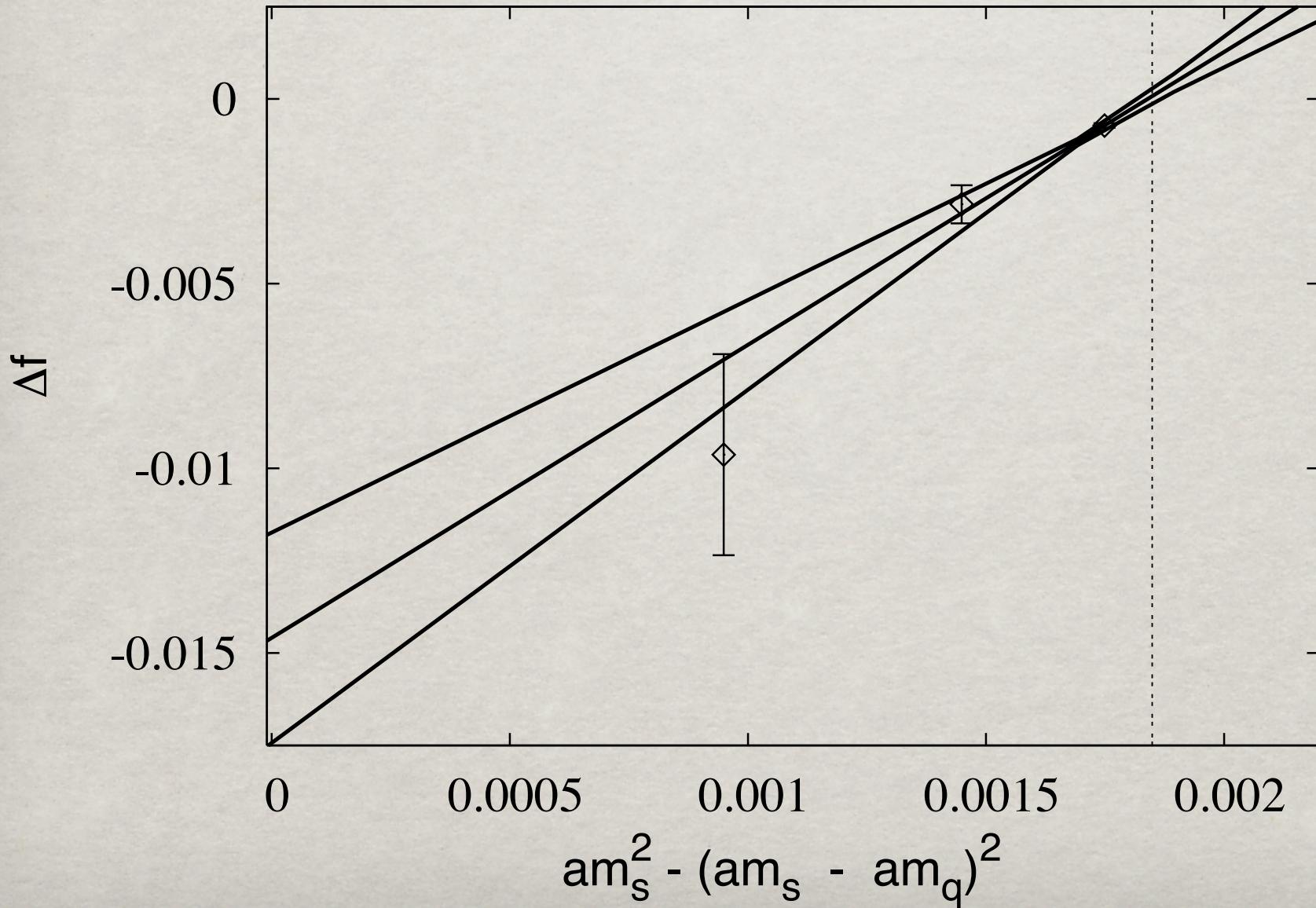
$$\Delta f = a + B(m_s - m_{ud})^2$$

$$R_{\Delta f} = \frac{\Delta f}{(M_K^2 - M_\pi^2)^2} = a + b(M_K^2 + M_\pi^2)$$

# CHIRAL EXTRAPOLATION OF $f_+(0)$

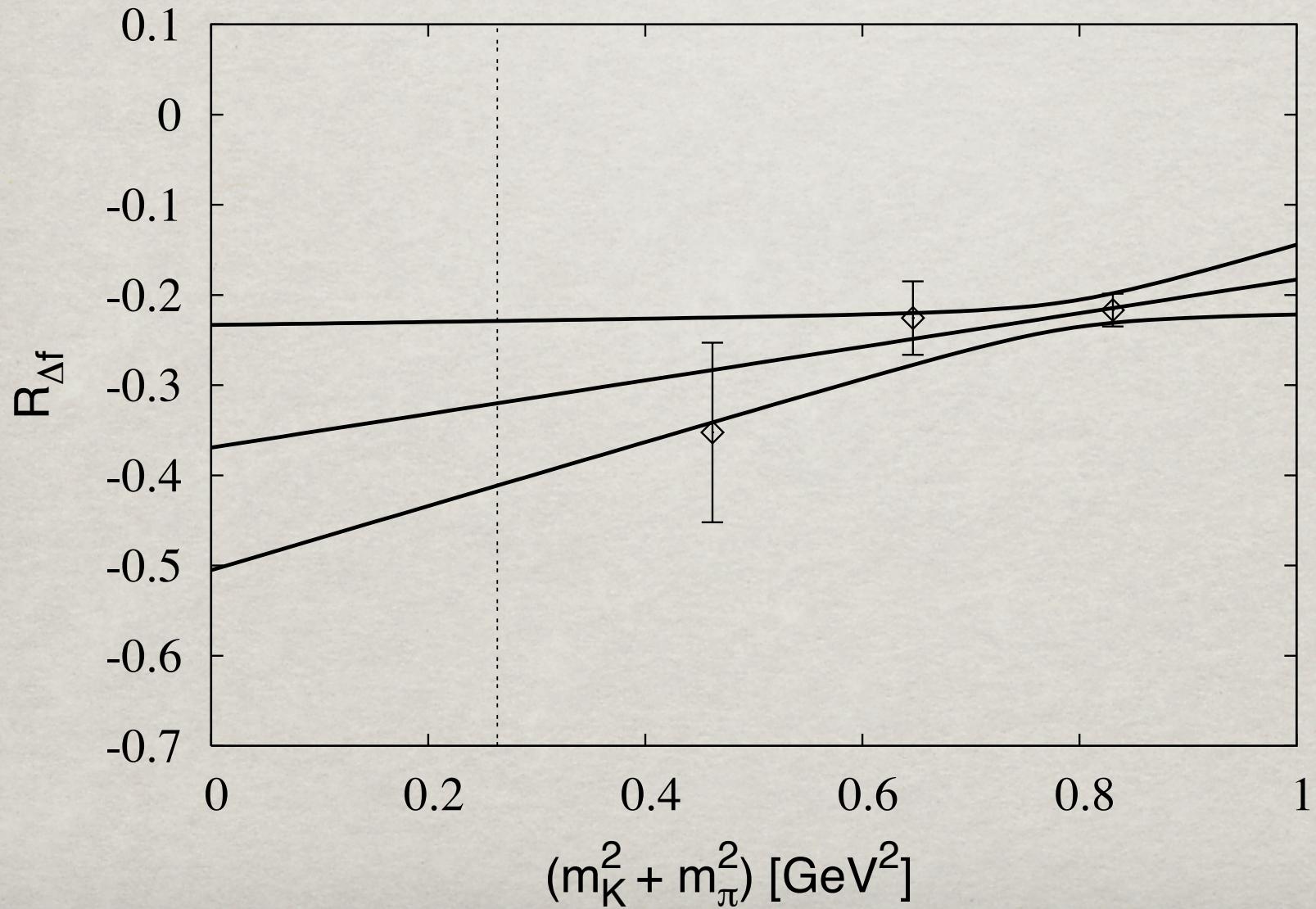
$$\Delta f = a + B(m_s - m_{ud})^2$$

$$\Delta f = -0.0146(28)$$



# CHIRAL EXTRAPOLATION OF $f_+(0)$

$$R_{\Delta f} = \frac{\Delta f}{(M_K^2 - M_\pi^2)^2} = a + b(M_K^2 + M_\pi^2) \quad \Delta f = -0.0161(46)$$



# ALTERNATIVE FITS TO $f_0(q^2)$

1. linear:

$$f_0(q^2) = f_0(0) + a_1 q^2$$

2. quadratic:

$$f_0(q^2) = f_0(0) + a_1 q^2 + a_2 q^4$$

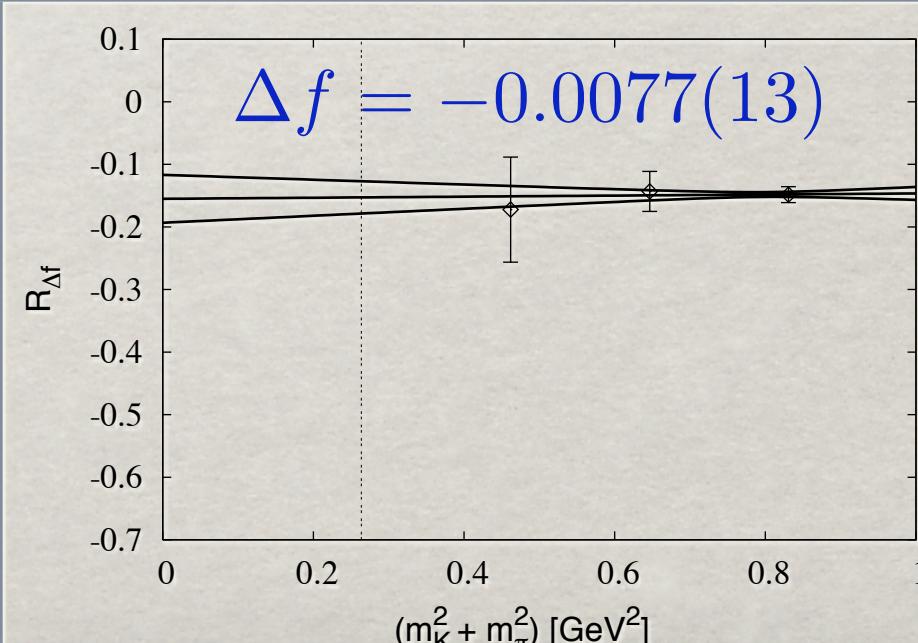
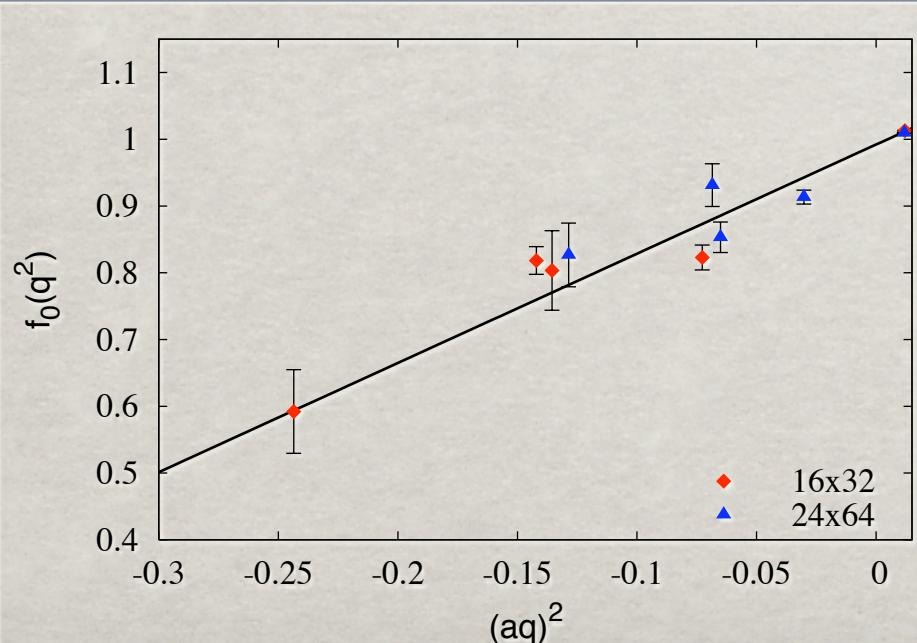
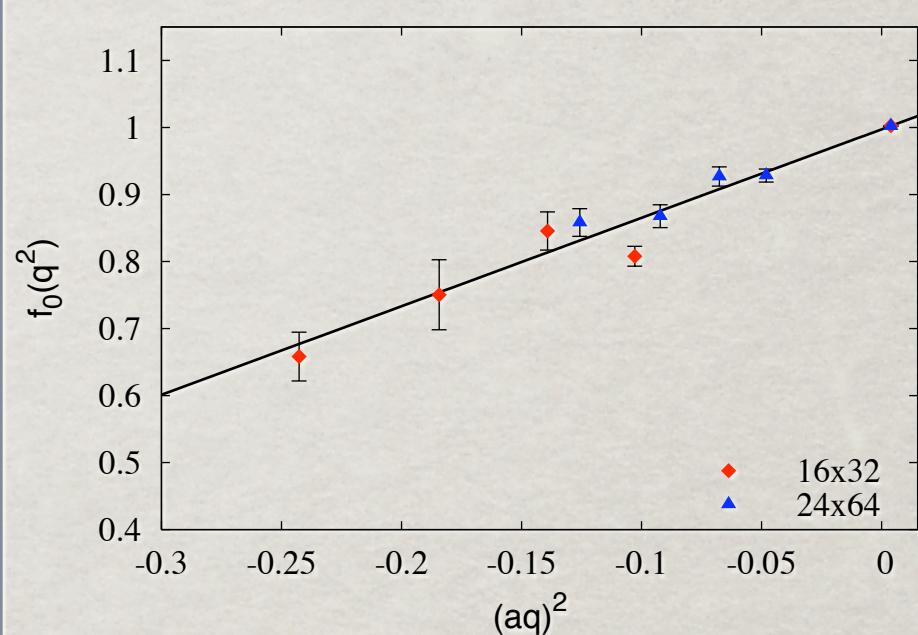
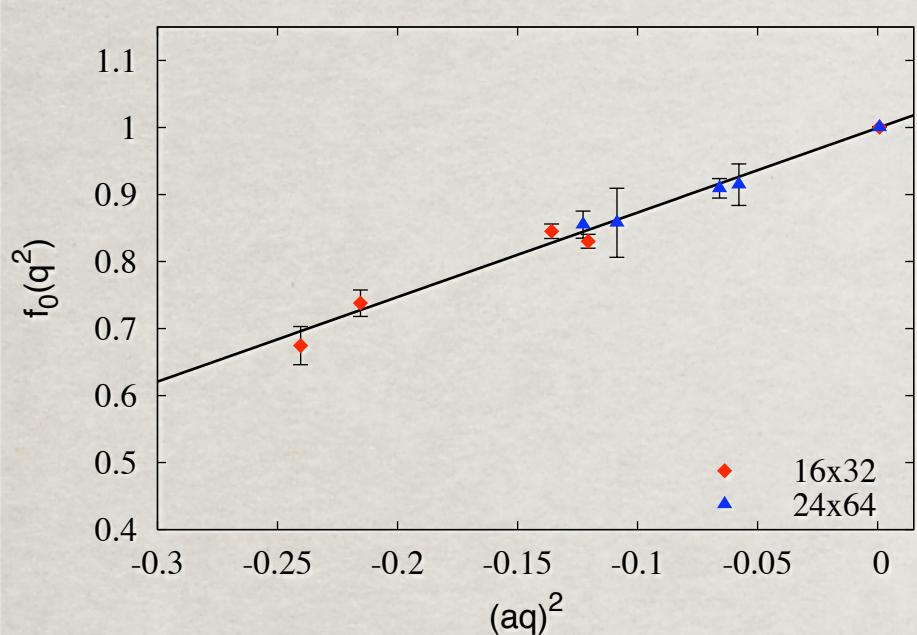
3. z-fit [hep-ph/0607108]:

$$f_0(t) = \frac{1}{\phi(t, t_0, Q^2)} \sum_{k=0}^{\infty} a_k(t_0, Q^2) z(t, t_0)^k$$
$$t_{\pm} \equiv (m_K \pm m_\pi)^2$$

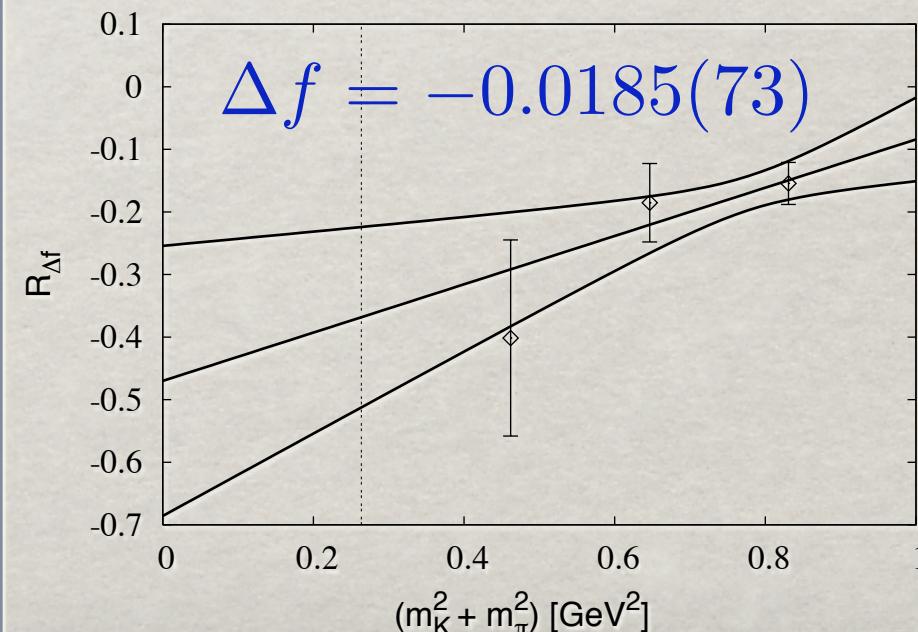
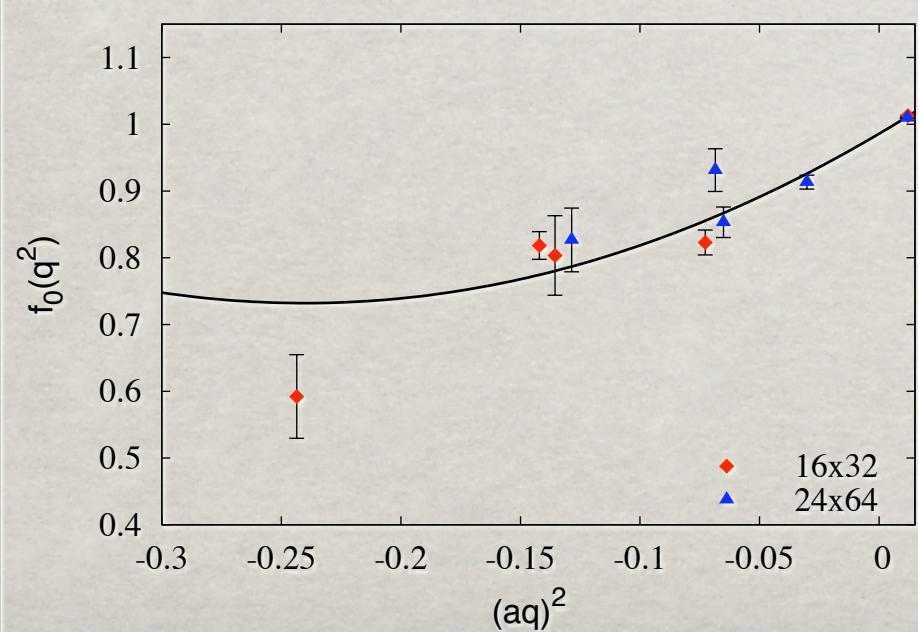
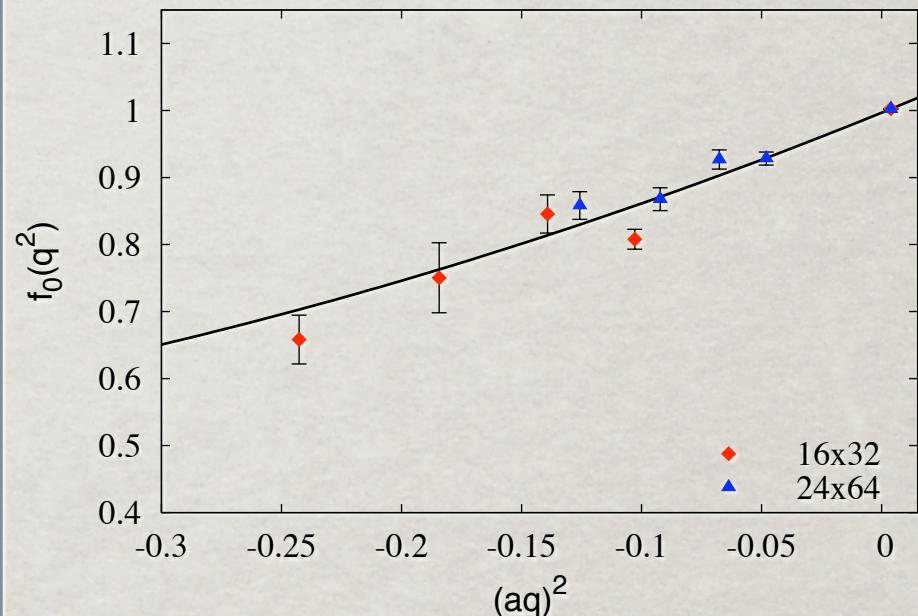
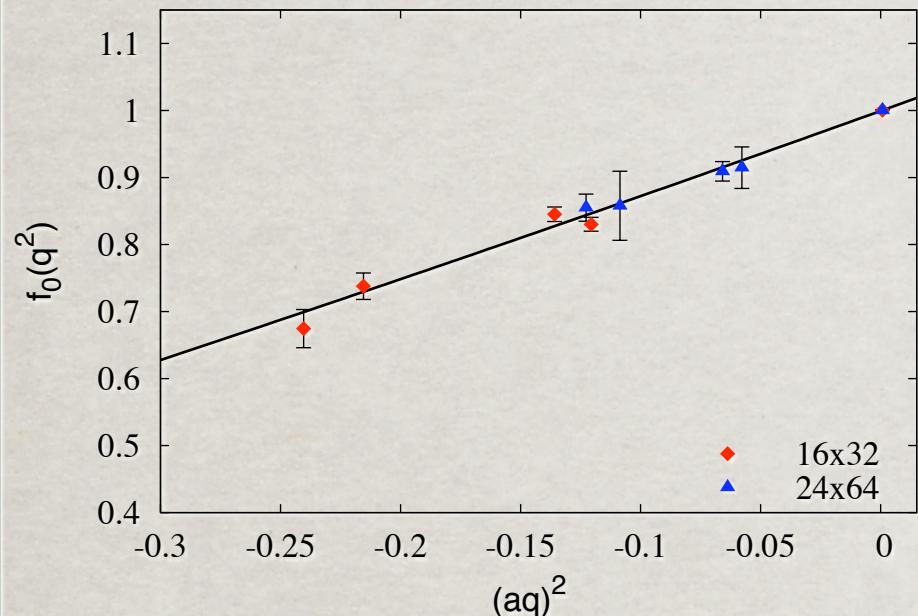
$$t = q^2 \rightarrow z(t, t_0) \equiv \frac{\sqrt{t_+ - t} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - t} + \sqrt{t_+ - t_0}}$$
$$t_0 \in (-\infty, t_+)$$
$$t_0 = t_+ (1 - \sqrt{1 - t_- / t_+})$$

$$\phi(t, t_0, Q^2) = \sqrt{\frac{3t_+ t_-}{32\pi}} \frac{z(t, 0)}{-t} \frac{z(t, -Q^2)}{-Q^2 - t} \left(\frac{z(t, t_0)}{t_0 - t}\right)^{-1/2} \left(\frac{z(t, t_-)}{t_- - t}\right)^{-1/4} \frac{\sqrt{t_+ - t}}{(t_+ - t_0)^{1/4}}$$

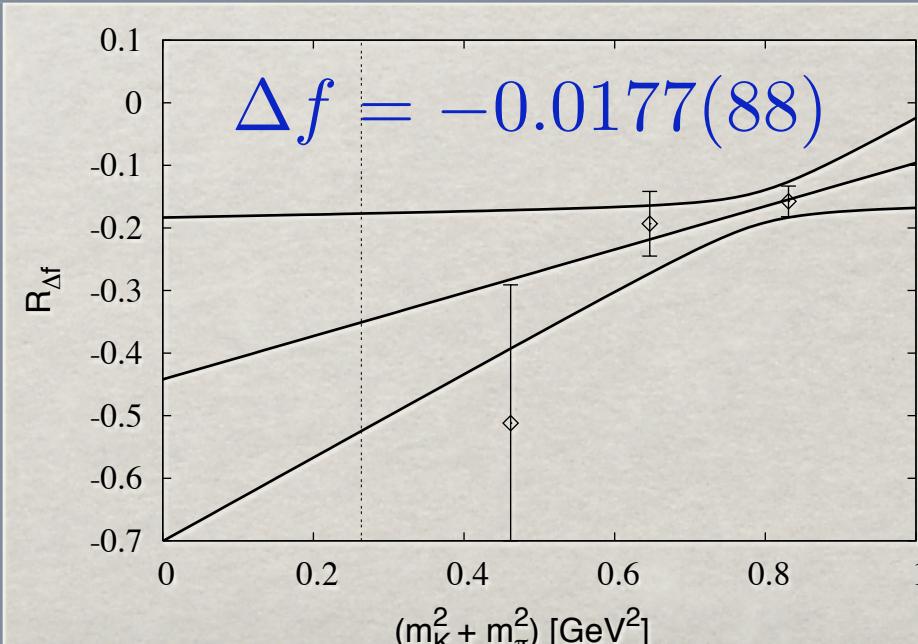
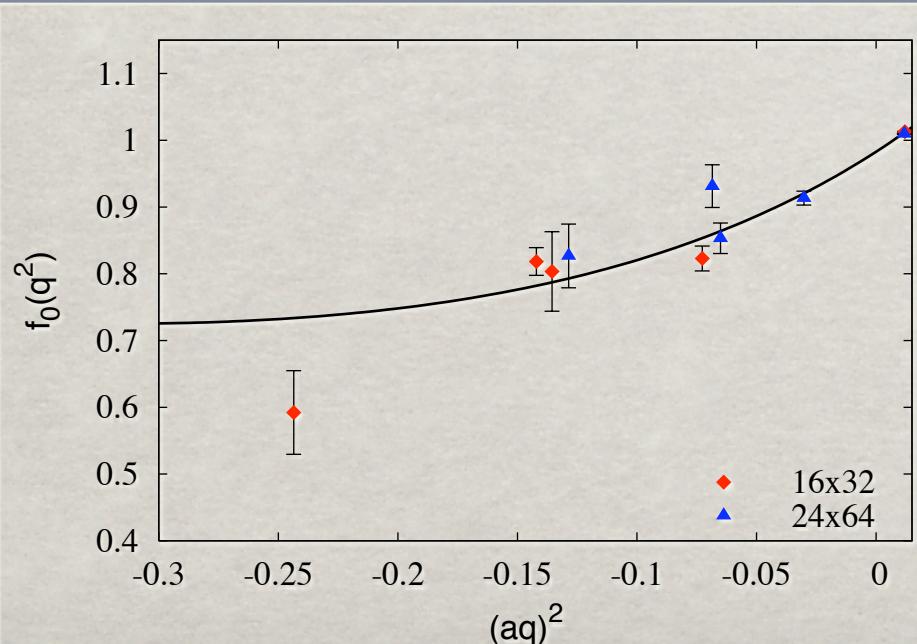
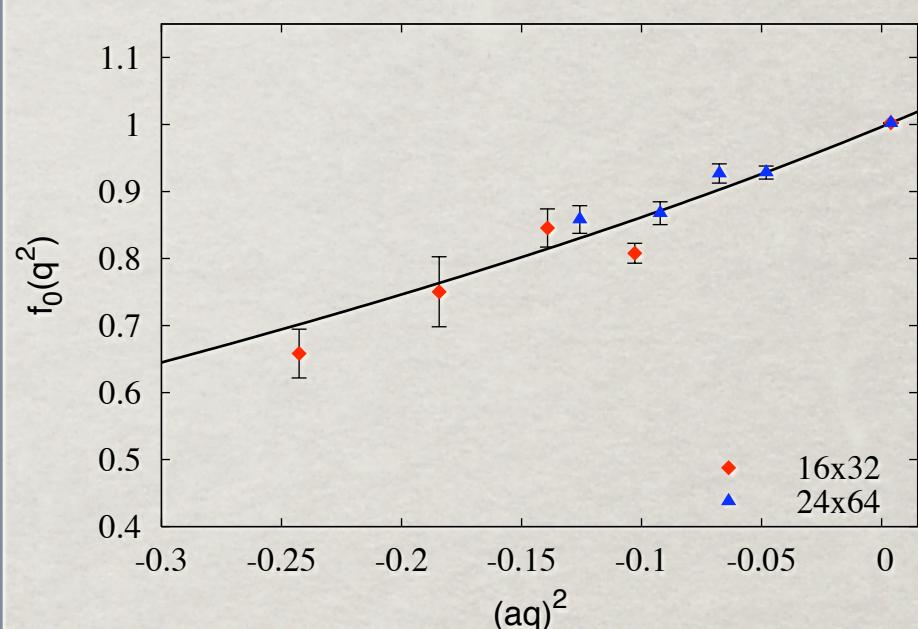
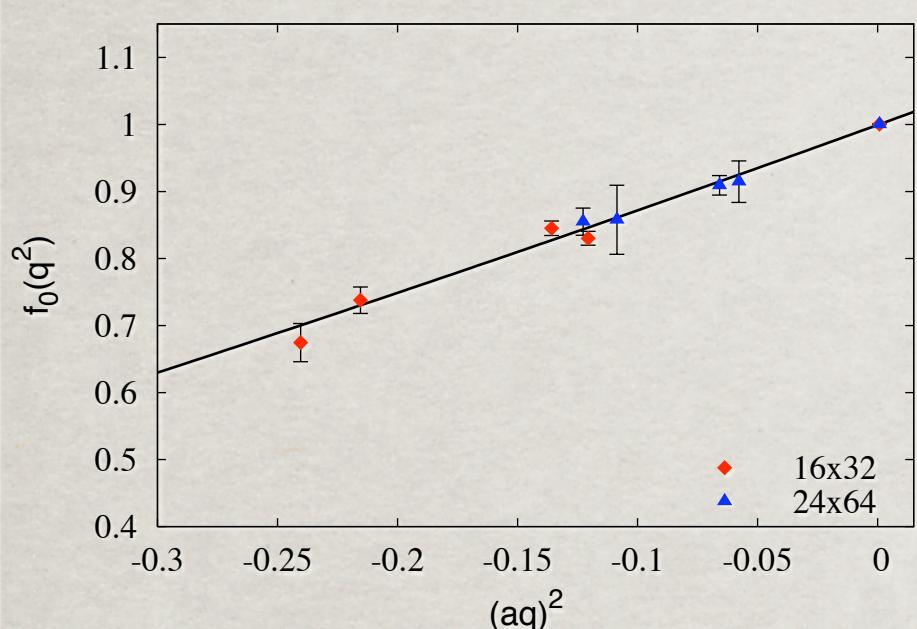
# 1. LINEAR: $f_0(q^2) = f_0(0) + a_1 q^2$



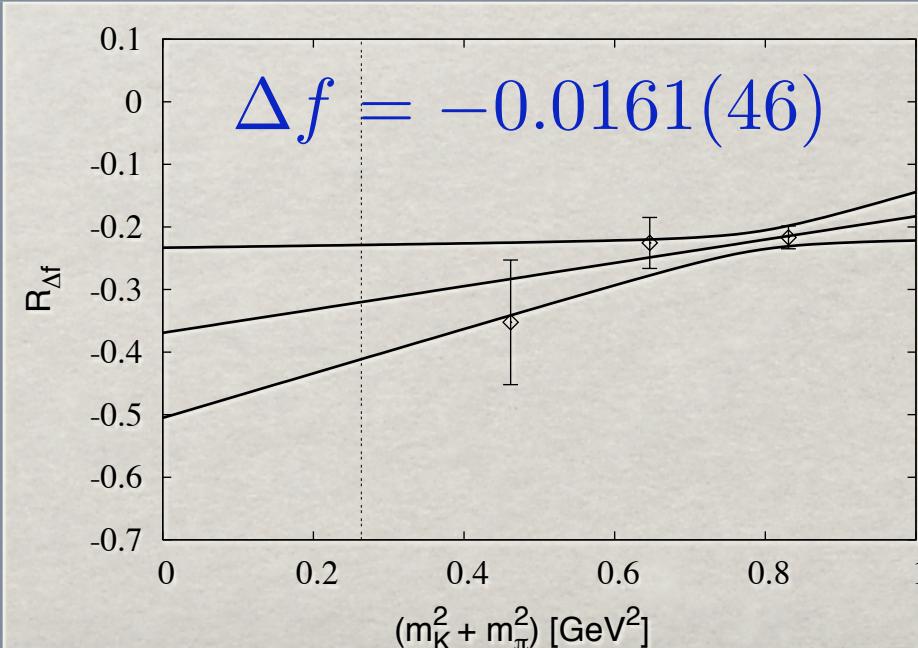
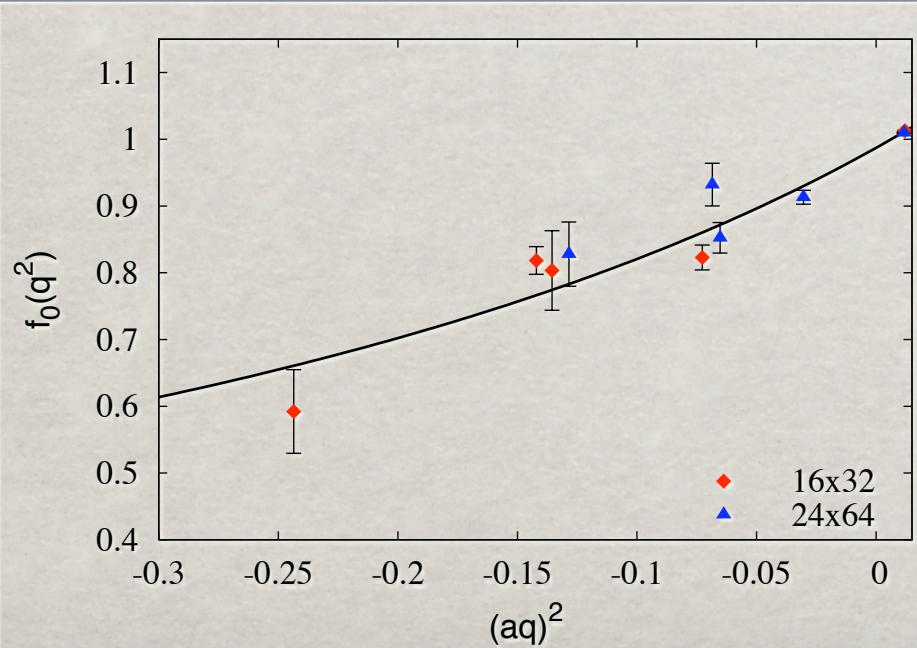
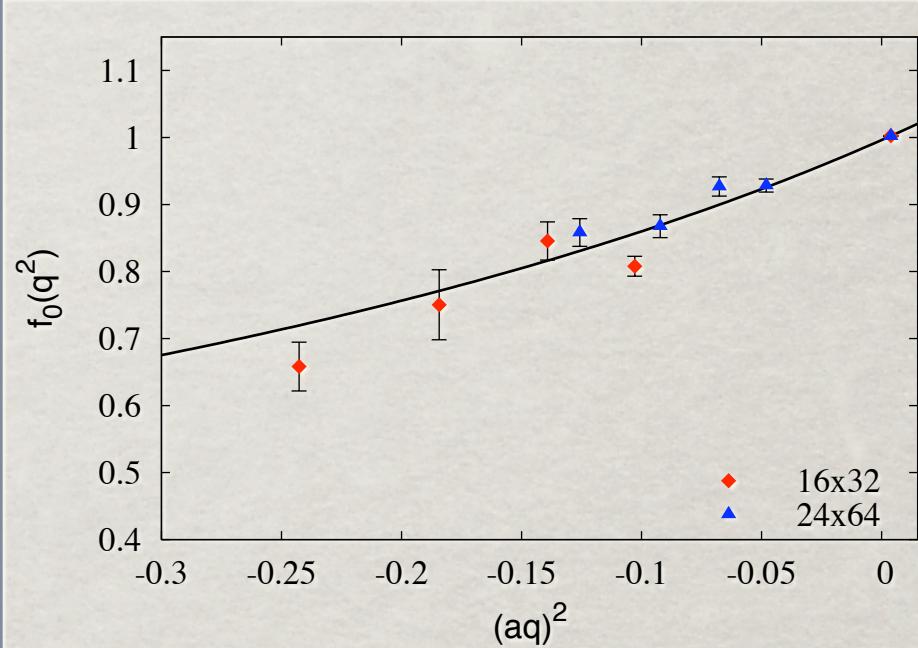
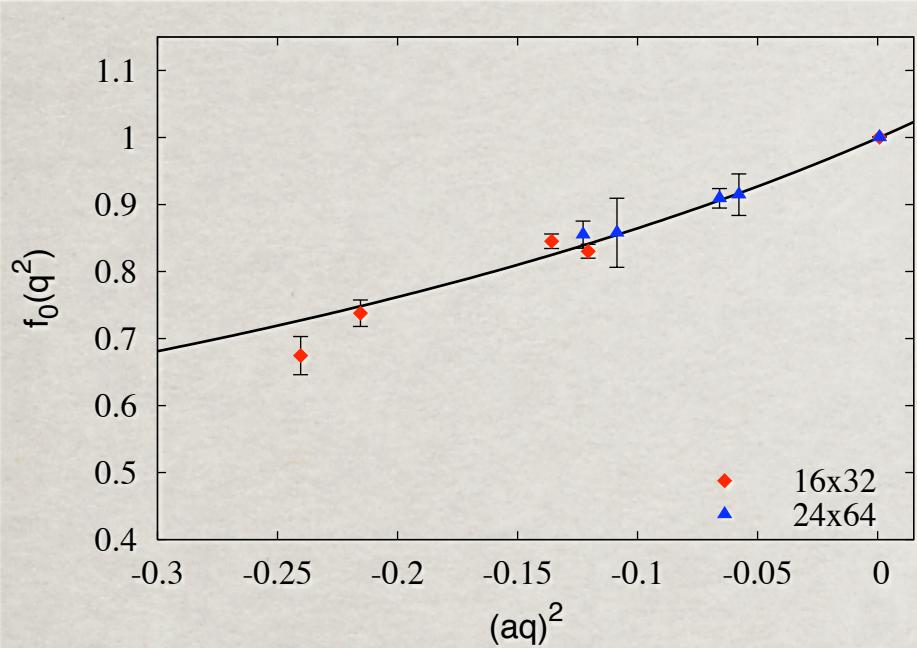
## 2. QUADRATIC: $f_0(q^2) = f_0(0) + a_1 q^2 + a_2 q^4$



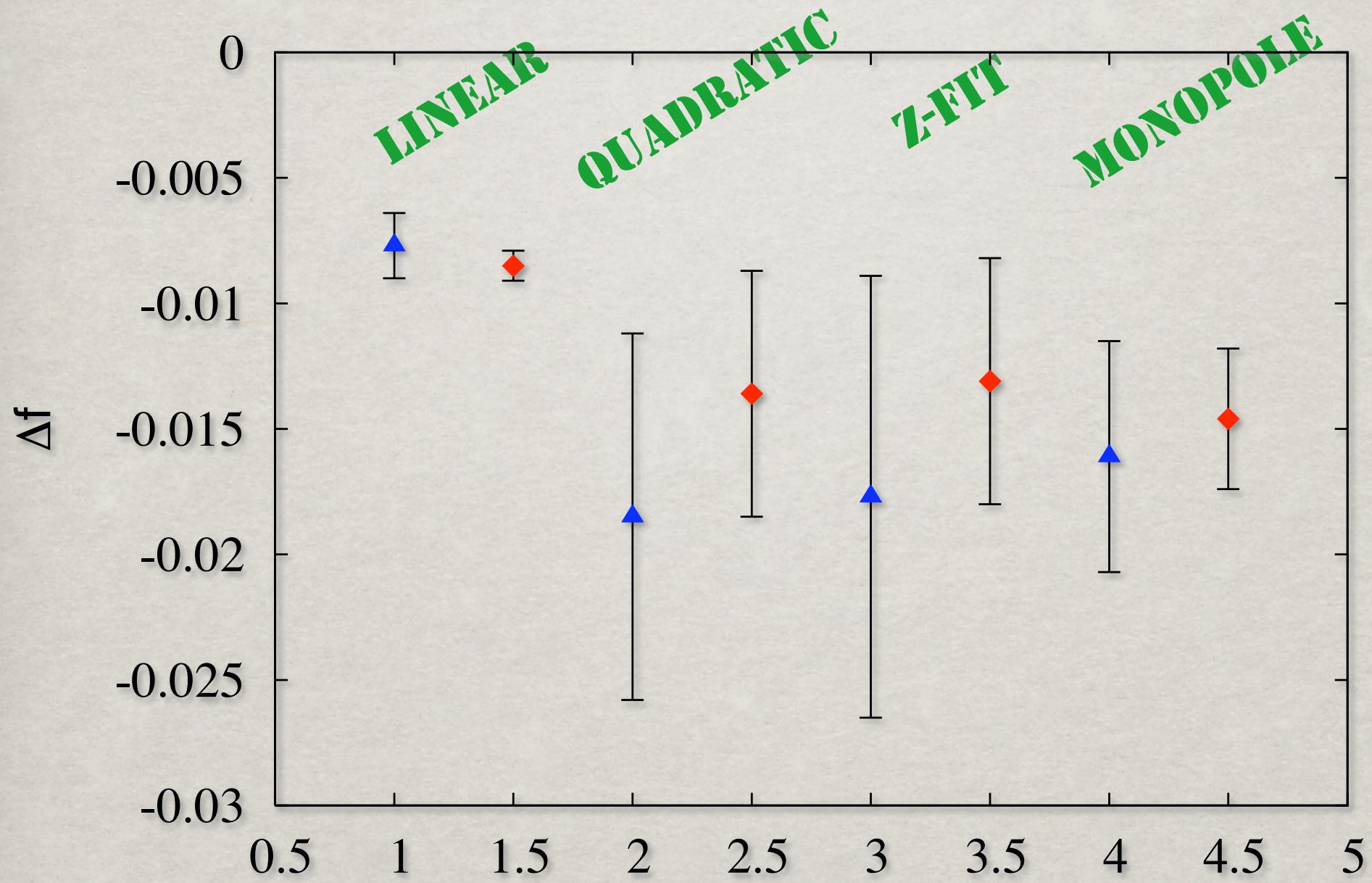
$$3. \text{ Z FIT: } f_0(t) = \frac{1}{\phi(t, t_0, Q^2)} \sum_{k=0}^{\infty} a_k(t_0, Q^2) z(t, t_0)^k$$



# 4. POLE: $f_0(q^2) = f_0(0)/(1 - q^2/M^2)$



# COMPARISON



$$|V_{us}|$$

$$\Delta f = -0.0161(46)(15)(16) \Rightarrow f_+^{K\pi}(0) = 0.9609(51)$$

Using  $|V_{us}f_+(0)| = 0.2169(9)$  from experimental decay rate:

$$|V_{us}| = 0.2257(9)_{\text{exp}}(12)_{f_+(0)}$$

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 - \delta, \quad \delta = 0.00076(62)$$

PDG(2006)/LR:

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 - \delta, \quad \delta = 0.0008(10)$$

# SUMMARY AND FUTURE WORK

Preliminary  $N_f = 2 + 1$  result for

- agrees well with L/R result
- no obvious finite size effects
- small statistical error
- progress towards controlling systematics

Further Improvements

- Lighter quark masses ( $am_q = 0.005$ )
- Another  $\beta \rightarrow$  continuum limit
- Twisted boundary conditions  $\rightarrow$  smaller  $q^2$